University of Toronto
Department of Mathematics
MAT 309F Introduction to Mathematical Logic
Fall, 1998
Homework 8. Sketch of Solution

Exercise 5.1.5: (1) Valid application of Substitution, since the term c is substitutable for the variable x_2 in the formula $R(x_1, x_2)$ and

$$R(x_1, x_2)[x_2/c] = R(x_1, c).$$

(2) (Done in class.) Valid application of Substitution, since the term x_1 is substitutable for the variable x_2 in the formula $Q(x_1, x_2) \leftrightarrow R(x_1, x_1)$ and

$$\big[Q(x_1, x_2) \leftrightarrow R(x_1, x_1)\big][x_2/x_1] = Q(x_1, x_1) \leftrightarrow R(x_1, x_1).$$

(3) Not a valid Application of Substitution because the term x_1 is not substitutable for the variable x_2 in the formula $\forall x_1 Q(x_2, c)$.

Exercise 5.1.6: (The first part of this exercise was done in class.) Step (3) is not valid because the term x_2 is not substitutable for the variable x_1 in the formula $\neg R(x_1) \vee \forall x_2 R(x_2)$.

Now, the formula $R(x_2) \rightarrow \forall x_2 R(x_2)$ is not valid. Let $\mathcal{A}$ be the structure $(\{a, b\}, R)$ where R is the monadic relation "$x = a$". Then $\mathcal{A} \models R(a)$, but $\mathcal{A} \not\models \forall x_2 R(x_2)$.

Exercise 5.1.7:

(1)

(i)	$\forall x \forall y \big[R(x, y) \rightarrow \neg R(y, x)\big]$	
(ii)	$\forall y \big[R(x, y) \rightarrow \neg R(y, x)\big]$	$\forall$-Elimination
(iii)	$R(x, x) \rightarrow \neg R(x, x)$	$\forall$-Elimination
(iv)	$\neg R(x, x)$	Propositional Logic
(v)	$\neg \forall x(x = x) \vee \neg R(x, x)$	Propositional Logic
(vi)	$\neg \forall x(x = x) \vee \forall x \neg R(x, x)$	Add $\forall$
(vii)	$\forall x(x = x)$	Reflexive Axiom
(viii)	$\forall x \neg R(x, x)$	Propositional Logic

(2) Let us first show that

(*) $$\vdash \forall x \forall y \neg R(x, y) \rightarrow \forall x \neg R(x, c).$$

By the Deduction Theorem, this is equivalent to showing that.

$$\forall x \forall y \neg R(x, y) \vdash \forall x \neg R(x, c).$$

(i)	$\forall x \forall y \neg R(x, y)$	
(ii)	$\neg R(b, c)$	$\forall$-Elimination (two times)
(iii)	$\neg \forall x(x = x) \vee \neg R(b, c)$	Propositional Logic

(iv) $\neg\forall x(x = x) \vee \forall x \neg R(x, c)$ Add $\forall$

(v) $\forall x(x = x)$ Reflexive Axiom

(vi) $\forall x \neg R(x, c)$ Propositional logic

Now,using $(*)$ and the definition of $\forall$, we get

$$\vdash \neg\exists x \exists y R(x, y) \rightarrow \neg\exists x \neg\neg R(x, c)$$

and then, using propositional logic, we obtain

$$\vdash \exists x R(x, c) \rightarrow \exists x \exists y R(x, y).$$

(3) By the Deduction Theorem, we just have to show that

$$\vdash R(b, c) \rightarrow \exists x \exists y R(x, y).$$

But this follows immediately by two applocations of Existential Generalization.

Exercise 5.1.12:

(1)

(i) $\forall x(x = x)$ Reflexive Axiom

(ii) $c = c$ $\forall$-Elimination

(iii) $\exists x(x = c)$ Existential Generalization

(iv) $\exists x[F(x) = c] \rightarrow \exists x(x = c)$ Propositional Logic

(2) Let $\mathcal{A}$ be the structure $(\,\{b, c\}, F, c\,)$, where F is the monadic function defined by

$$F(b) = c$$
$$F(c) = b.$$

Then, $\mathcal{A} \models \exists x(x = c)$, but $\mathcal{A} \not\models \exists x[F(x) = c]$.