University of Toronto
Department of Mathematics
MAT 309F Introduction to Mathematical Logic
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Homework 7. Sketch of Solution

Note. Below, we use the notational conventions that we have used in class.

- If $\mathcal{A}$ is a structure, $A(x)$ is a formula in language appropriate for $\mathcal{A}$, and a is an element of the universe of $\mathcal{A}$, we write $A(a)$ instead of the more precise (but notationally cumbersome) notation $A[x/c_a]$ (where c_a is a name for the element a).
- Formally, we should use the symbols $x_1, x_2, \ldots$ to denote variables. Again, we sacrifice formality for readability, and instead use the last letters of the Roman alphabet in lowercase x, y, z.

Exercise 4.13.13: (1) We have to prove that for every structure $\mathcal{A}$,

$$\mathcal{A} \models (\forall x A(x) \vee \forall x B(x)) \to \forall x (A(x) \vee B(x)).$$

By the inductive definition of $\models$, this is equivalent to showing that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \forall x A(x) \vee \forall x B(x) \quad \text{implies} \quad \mathcal{A} \models \forall x (A(x) \vee B(x)).$$

Suppose $\mathcal{A} \models \forall x A(x) \vee \forall x B(x)$. By the inductive definition of $\models$, we have two cases.
Case 1: $\mathcal{A} \models \forall x A(x)$. Then, if a is an arbitrary element of the universe of $\mathcal{A}$, we have $\mathcal{A} \models A(a)$ and hence $\mathcal{A} \models A(a) \vee B(a)$. Since a is arbitrary, the inductive definition of $\models$ allows us to conclude $\mathcal{A} \models \forall x (A(x) \vee B(x))$.
Case 2: $\mathcal{A} \models \forall x B(x)$. Then, if a is an arbitrary element of the universe of $\mathcal{A}$, we have $\mathcal{A} \models B(a)$ and hence $\mathcal{A} \models A(a) \vee B(a)$. Since a is arbitrary, by the inductive definition of $\models$ we can conclude $\mathcal{A} \models \forall x (A(x) \vee B(x))$.

(2) Let a, b be distinct objects, and let $\mathcal{A} = (\,\{a, b\}, a, b\,)$ (*i.e.*, the universe of $\mathcal{A}$ is the set $\{a, b\}$, and a, b are regarded as constants). Then

$$\mathcal{A} \models \forall x (x = a \vee x = b) \quad \text{but} \quad \mathcal{A} \not\models \forall x (x = a) \vee \forall x (x = b).$$

(3) We have to prove that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \exists x (A(x) \wedge B(x)) \to \exists x A(x) \wedge \exists x B(x).$$

By the inductive definition of $\models$, this is equivalent to showing that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \exists x(A(x) \wedge B(x)) \quad \text{implies} \quad \models \exists x A(x) \wedge \exists x B(x).$$

Suppose $\mathcal{A} \models \exists x(A(x) \wedge B(x))$. Then there exists an element a in the universe of $\mathcal{A}$ such that $\mathcal{A} \models A(a) \wedge B(a)$. We have $\mathcal{A} \models A(a)$ and $\mathcal{A} \models B(a)$. Hence, a witnesses the fact that $\mathcal{A} \models \exists x A(x)$ and $\mathcal{A} \models \exists x B(x)$. From these two facts and the inductive definition of $\models$, we conclude $\mathcal{A} \models \exists x A(x) \wedge \exists x B(x)$.

(4) Let $\mathcal{A}$ be the same structure as in (2). Then

$$\mathcal{A} \models \exists x(x = a) \wedge \exists x(x = b) \quad \text{but} \quad \mathcal{A} \not\models \exists x(x = a \wedge x = b).$$

(5) We have to prove that for every structure $\mathcal{A}$,

$$\mathcal{A} \models (\exists x A(x) \vee B) \to \exists x(A(x) \vee B).$$

By the inductive definition of $\models$, this is equivalent to showing that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \exists x A(x) \vee B \quad \text{implies} \quad \mathcal{A} \models \exists x(A(x) \vee B).$$

Suppose $\mathcal{A} \models \exists x A(x) \vee B$. By the inductive definition of $\models$, we have two cases.

Case 1: $\mathcal{A} \models \exists A(x)$. By definition, there exists an element a of the universe of $\mathcal{A}$ such that $\mathcal{A} \models A(a)$. We then have $\mathcal{A} \models A(a) \vee B$. But then the element a witnesses the fact that $\mathcal{A} \models \exists x(A(x) \vee B(x))$.

Case 2: $\mathcal{A} \models B$. Then, If a is an arbitrary element of the universe of $\mathcal{A}$, we have $\mathcal{A} \models A(a) \vee B$. Thus, the element a witnesses the fact that $\mathcal{A} \models \exists x(A(x) \vee B)$.

(6) Let $\mathcal{A}$ be the structure given in (2). Then

$$\mathcal{A} \models \exists x(\neg(x = x) \vee x = a).$$

However

$$\mathcal{A} \not\models \exists x \neg(x = x) \vee x = a.$$

(7) We have to prove that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \forall x(A(x) \wedge B(x)) \quad \text{implies} \quad \mathcal{A} \models \forall x A(x) \wedge B(x).$$

Assume $\mathcal{A} \models \forall x(A(x) \wedge B(x))$. Fix an arbitrary a in the universe of $\mathcal{A}$. Then
(i) $\mathcal{A} \models A(a)$,
(ii) $\mathcal{A} \models B(a)$.
Now, since a is arbitrary, (i) implies that $\mathcal{A} \models \forall x A(x)$. Similarly, since a is arbitrary, (ii) implies that $\mathcal{A} \models B$. Hence, $\mathcal{A} \models \forall x A(x) \wedge B$.

(8) This formula *is* logically valid. The book has an error here. The details were discussed in class.

Exercise 4.13.15: (1) By the inductive definition of $\models$, we have to show that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \forall x(A(x) \to B(x)) \quad \text{iff} \quad \models \neg\exists x(A(x) \wedge \neg B(x)).$$

($\Rightarrow$): Assume that $\models \exists x(A(x) \wedge \neg B(x))$. Then there exists an element a in the universe of $\mathcal{A}$ such that $\mathcal{A} \models A(a)$ and $\mathcal{A} \models \neg B(a)$. Then $\mathcal{A} \not\models B(a)$, so a witnesses the fact that we cannot have $\mathcal{A} \models \forall x(A(x) \to B(x))$.

($\Leftarrow$): Assume that $\mathcal{A} \not\models \forall x(A(x) \to B(x))$. Then there must be an element a in the universe of $\mathcal{A}$ such that $\mathcal{A} \not\models A(a) \to B(a)$. By the definition of $\models$, for this a we have $\mathcal{A} \models A(a)$ and $\mathcal{A} \models \neg B(a)$. Thus, a witnesses the fact that $\mathcal{A} \models \exists x(A(x) \wedge \neg B(x))$, contradicting our assumption.

(2) We now have to show that for every structure $\mathcal{A}$,

$$\mathcal{A} \models \forall x(A(x) \to B(x)) \quad \text{iff} \quad \models \neg\exists x(A(x) \wedge \neg B(x)).$$

($\Rightarrow$): Assume that $\mathcal{A} \not\models \forall x(A(x) \to \neg B(x))$. Then there must be an element a in the universe of $\mathcal{A}$ such that $\mathcal{A} \not\models A(a) \to \neg B(a)$. This means that $\mathcal{A} \models A(a)$ and $\mathcal{A} \not\models \neg B(a)$, *i.e.*, $\mathcal{A} \models B(a)$. Thus, a witnesses the fact that $\mathcal{A} \models \exists x(A(x) \wedge B(x))$, which contradicts the assumption that $\mathcal{A} \not\models \exists x(A(x) \wedge B(x))$.

($\Leftarrow$): Assume $\models \exists x(A(x) \wedge B(x))$. Then there exists an element a in the universe of $\mathcal{A}$ such that $\mathcal{A} \models A(a)$ and $\mathcal{A} \models B(b)$, *i.e.*, $\mathcal{A} \not\models \neg B(b)$. Thus, $\mathcal{A} \not\models A(a) \to \neg B(a)$, so a witnesses the fact that we cannot have $\mathcal{A} \models \forall x(A(x) \to \neg B(x))$.