University of Toronto
Department of Mathematics
MAT 309F Introduction to Mathematical Logic
Fall, 1998
Homework 1. Sketch of Solution

Exercise 1.5.4: Let $A = \{0, 1\}$. Then A is countable. Let

$$A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$$

Then A^n is countable for every n since a cartesian product of countable sets is countable (proved in class). Now, since a countable union of countable sets is countable (also proved in class),

$$A^* = \bigcup_{n=0}^{\infty} A^n$$

is countable. But clearly, $A*$ is in bijective correspondence with the set of all finite sequences of 0's and 1's.

Exercise 1.5.9: Since A is countable, we can fix a bijection $h : A \to \mathbb{N}$. Let $g : B \to A$ be defined as in the hint, and notice that $f(g(b)) = b$ for every $b \in B$. The function g is one-to-one because $g(b) = g(b')$ implies $b = f(g(b)) = f(g(b')$. We have

$$B \approx g(B) \approx h(g(B))$$

and $h(g(B))$ is countable since it is a subset of $\mathbb{N}$. Hence B is countable.

Exercise 1.5.12: Suppose that A is countable, and let

$$A = \{a_0, a_1, a_2 \ldots\}.$$

Let the decimal expansion of a_n be

$$0.a_{n,0}a_{n,1}\ldots.$$

Define

$$b_n = \begin{cases} 2, & \text{if } a_{n,n} = 3 \\ 3, & \text{if } a_{n,n} = 2 \end{cases}$$

and let b be the number whose decimal expansion is $b_0, b_1, \ldots$. Then $b \in A$, but $b \neq a_n$ for every n because b and a_n differ in the nth decimal place. We have reached a contradiction. Hence A cannot be countable.

Exercise 1.5.15: Let

$$\mathbb{N}^n = \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ times}}$$

Then $\mathbb{N}^n$ is countable, since a cartesian product of countable sets is countable. Now, since a countable union of countable sets is countable,

$$\mathbb{N}^* = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

is countable. (Notice that we have used the same argument as in Exercise 1.5.4!) Define now a map g from $\mathbb{N}^*$ into the set of finite subsets of $\mathbb{N}$ by letting

$$g((k_1, \ldots, k_n)) = \{ k_1, \ldots, k_n \}.$$

The function g is onto, so, by Exercise 1.5.9, the set of finite subsets of $\mathbb{N}$ is countable.

Now the set of *all* subsets of $\mathbb{N}$ is not countable because it is in bijective correspondence with the set of infinite sequences of 0's and 1's (a 1 on the nth position means that n is in the set, a 0 on the nth position means that n is not in the set).

The set of all infinite subsets of $\mathbb{N}$ cannot be countable because otherwise, its union with the set of finite subsets would be countable. But we have shown in the preceding paragraph that this union is uncountable.