University of Toronto
Department of Mathematics
MAT 334F Complex Variables
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Homework 3. Sketch of Solution

Exercise 5: Let us first solve the equation geometrically and then algebraically.

Geometric solution. The set of points

$$e^{i\theta}, \qquad 0 \leq \theta < 2\pi,$$

is the unit circle. Therefore, what we want to find is the set of points on the unit circle whose distance to 1 equals 2. Well, there is only one such point, namely, -1. Thus, the only solution is $\theta = \pi$.

Algebraic solution. Set $|e^{i\theta} - 1| = 2$. Then

$$\begin{aligned} 4 &= (\cos\theta - 1)^2 + (\sin\theta)^2 \\ &= (\cos\theta - 1)^2 + 1 - (\cos\theta)^2 \\ &= 2\cos\theta + 2. \end{aligned}$$

Now, since $-1 \leq \cos\theta \leq 1$, the only solution is $2\cos\theta = 2$, i.e., $\theta = \pi$.

Exercise 12: Part (a) is immediate from the recipe derived in class to compute all the nth roots of a complex number (where n is any positive integer). We answer part (b)

Notice that $\sqrt{A}\cos\alpha = \mathrm{Re}(a + i) = a$. Therefore,

$$\begin{aligned} \pm\sqrt{A}\exp(i\alpha/2) &= \pm\sqrt{A}\Big(\frac{1+\cos\alpha}{2} + i\frac{1-\cos\alpha}{2}\Big) \\ &= \pm\Big(\frac{\sqrt{A}+\sqrt{A}\cos\alpha}{2} + i\frac{\sqrt{A}-\sqrt{A}\cos\alpha}{2}\Big) \\ &= \pm\frac{1}{\sqrt{2}}\big(\sqrt{A+a} + i\sqrt{A-a}\big). \end{aligned}$$

Exercise 14: The fourth roots of -4 are

$$\omega_1 = \sqrt{2}e^{i\frac{\pi}{4}}, \qquad \omega_2 = \sqrt{2}e^{i\frac{3\pi}{4}}, \qquad \omega_3 = \sqrt{2}e^{i\frac{5\pi}{4}}, \qquad \omega_4 = \sqrt{2}e^{i\frac{7\pi}{4}}.$$

Hence,

$$\begin{aligned}z^4+4 &= (z-\omega_1)(z-\omega_2)(z-\omega_3)(z-\omega_4)\\ &= \big[(z-\omega_1)(z-\omega_4)\big]\big[(z-\omega_2)(z-\omega_3)\big]\\ &= \big[z^2+(\omega_1+\omega_4)z+\omega_1\omega_4\big]\big[z^2+(\omega_2+\omega_3)z+\omega_2\omega_3\big]\\ &= \big[z^2+2z+2\big]\big[z^2-2z+2\big].\end{aligned}$$

Exercise 20: (a) Let

$$\begin{aligned}\alpha &= \frac{-b+\sqrt{b^2-4ac}}{2a}\\ \beta &= \frac{-b-\sqrt{b^2-4ac}}{2a}.\end{aligned}$$

Then

$$\begin{aligned}-\alpha-\beta &= \frac{b}{a}\\ \alpha\beta &= \frac{b^2-(b^2-4ac)}{4a} = \frac{c}{a}.\end{aligned}$$

Therefore,

$$\begin{aligned}z^2+bz+c &= \frac{1}{a}\Big(z^2+\frac{b}{a}z+\frac{c}{a}\Big)\\ &= \frac{1}{a}(z-\alpha)(z-\beta),\end{aligned}$$

and hence $z^2+bz+c=0$ if and only if $z=\alpha$ or $z=\beta$.

(b) By part (a), the roots of the equation are given by

$$z = \frac{-2+(2^2-4(1-i))^{1/2}}{2}.$$

Hence, the two roots of the equation are

$$\frac{-2+\sqrt{2}(1+i)}{2}, \qquad \frac{-2-\sqrt{2}(1+i)}{2}.$$